Ref (15)
1979

PROXIMITY

by

A.G. PORTELA

May 8th. 1979

ABSTRACT

"Distance", in a conversational context, has not the same strict meaning used in mathematical language.
Some kind of a "fuzzy-distance" should be formerly defined.
The word "fuzzy" being so much connected to fuzzy-sets, it was deemed convenient to coin a new word, namely Proximity (or PROX).
Various types or classes of Proximities can be introduced.
Usually some kind of "triangular law" is imposed in many classes of Prox.
Proximities have various applications namely in graphs and hypergraphs, thermodynamics, mechanics, physics, etc..

RÉSUMÉ

Distance, dans un contexte conversationnel, n'a pas la même stricte signification employée en langage mathématique.
Une certaine forme de"distance-floue" devrait être définie. Le mot "flou", étant si connecté à la théorie des ensembles-flous, on a jugé utile introduire un vocable nouveau comme par exemple Proximité (ou PROX). Différentes classes de proximités peuvent être introduites.
Normalement, une certaine forme de "lois triangulaire" est imposée aux classes de proximités.
Les proximités ont différantes applications, comme par exemple: graphes et hypergraphes, thermodynamique, physique, mécanique, etc..

PROXIMITY

I) "Distance", in a conversational context, has not the same strict meaning used in mathematical language.
The "distance" between two cities is measured by the time it takes to travel or the price of the fare, etc..
These "distances" are functions of the locations of the cities considered, but they do not comply to the formal definition of distance.
Some kind of a "fuzzy-distance" should be formerly defined.
The word "fuzzy" being so much connected to fuzzy-sets, it was deemed convenient to coin a new word, namely Proximity (or PROX).
Various types or classes of Proximities can be introduced.
Usually some kind of "triangular law" is imposed in many classes of Prox.
Proximities have various applications namely in graphs and hypergraphs, thermodynamics, mechanics, physics etc..

II) Definition of Proximity

1st. Rule

A proximity is a functional relation of the type: $X^2 \xrightarrow{\mathcal{R}} \wp$ .....(1)

where:

X is a set or a Cartesean product of sets.
The set or sets or the Cartesean product can be algebraically and or topologically structured.

$$\text{dom } \mathcal{R} \equiv X^2$$

$\wp$ is a set usually algebraically and topologically structured.
The elements of $\wp$ are numbers, sets, distribuitions etc..

$$\text{Range } (\mathcal{R}) \equiv \wp$$

$\mathcal{R}$ is a functional relation:

$$(x_i, x_j) \xrightarrow{\mathcal{R}} q$$

$$\forall\ (x_i, x_j) \in X^2 \ ,\ q \in \wp$$

2nd. Rule

The relation $\mathcal{R}$ includes some kind of a triangular law, or an associated distance.

III) Examples of Proximities

The best way to present the concept of Proximity is by giving some typical examples.

A) Slack-distance

X is a vector space

$$\mathcal{G} \equiv \{ [h_1, h_2) : h_1, h_2 \in R^+ \}$$

$\mathcal{R}$ is functional: $(x_i, x_j) \xrightarrow{\mathcal{R}} [h_{1_{ij}}, h_{2_{ij}})$
$\forall (x_i, x_j) \in X^2$

and satisfies de following conditions:

a) $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i) = [h_{1_{ij}}, h_{2_{ij}})$

b) If
$$\mathcal{R}(x_i, x_j) = [h_{1_{ij}}, h_{2_{ij}}) \qquad (1)$$
$$\mathcal{R}(x_j, x_k) = [h_{1_{jk}}, h_{2_{jk}})$$
$$\mathcal{R}(x_i, x_k) = [h_{1_{ik}}, h_{2_{ik}})$$

than
$$h_{1_{ij}} + h_{1_{jk}} \geq h_{1_{ik}} \qquad (2)$$

c) and $(h_{2_{ij}} - h_{1_{ij}}) + (h_{2_{jk}} - h_{1_{jk}}) \geq (h_{2_{ik}} - h_{1_{ik}})$ (3)

d) $\mathcal{R}(x_i, x_k) = \phi, \quad \underset{\forall i, k}{x_i = x_k}$ (4)

The lower limits (h_1) are bound by an imposed distance, but the upper limits are more free, this enables the proximity just defined to accommodate some "slackness".

Other types of slack-distances can be conceived, as for instance:

a') $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i) = (h_{1_{ij}}, h_2]$ (5)

b') $h_{2_{ij}} + h_{2_{jk}} \geq h_{2_{ik}}$ (6)

c') $(h_{ij2} - h_{ij1}) + (h_{jk2} - h_{jk1}) \geqslant (h_{ik2} - h_{ik1})$ (7)

d') $\mathcal{R}(x_i, x_k) = \phi$, if $x_i = x_4, \forall_{i,4}$ (8)

or

a'') $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i) = (h_{ij1}, h_{ij2})$ (9)

b'') $h_{ij} + h_{jk} \geqslant h_{ik}$ (10)

c'') $(h_{ij2} - h_{ij1}) + (h_{jk2} - h_{jk1}) \geqslant (h_{ik2} - h_{ik1})$ (11)

d'') $\mathcal{R}(x_i, x_k) = \phi$ if $x_i = x_k, \forall_{ik}$ (12)

where

$$h_{ij} \in (h_{ij1}, h_{ij2})$$
$$h_{jk} \in (h_{jk1}, h_{jk2})$$
$$h_{ik} \in (h_{ik1}, h_{ik2})$$

or

a''') $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i) = (h_1, \infty)$ (13)

b''') $h_{ij1} + h_{jk1} \geqslant h_{ik1}$ (14)

c''') is deleted

d''') $\mathcal{R}(x_i, x_j) = \phi$ if $x_i = x_j \ \forall_{ij}$ (15)

<u>Note 1:</u> If $h_2 - h_1 \longrightarrow 0$, the slack-distance tends to a normal distance.
If $h^* = \mathrm{Lim}\, h_1 = \mathrm{Lim}\, h_2$, when $h_2 - h_i \longrightarrow 0$
$h^* \in R^+$ and $\wp \equiv R^+$, than a distance in the mathematical sense, is obtained.

<u>Note 2:</u> All slack-distances include a distance either on the lower limit, on the upper limit or in the interval.

<u>Note 3:</u> A condition e) could be added, namely:

e) $x_i = x_j$ if $\mathcal{R}(x_i, x_j) = \phi$ (16)

B) Proximities using fuzzy-sets

If $\wp$ is the universal set and $\Gamma_A(\alpha)$ is the membership function of a fuzzy-set A of $\wp$ and $\alpha \in \wp$, another kind of "proximities" can be introduced, as for instance:

a) $\mathcal{R}(x_i, x_j) = \Gamma_A(\alpha) \qquad \forall x_i, x_j \in X$ (17)
and $x_i \neq x_j$

b) Zadeh fuzzy-sets are used:

$$\alpha \in \wp \equiv R^+$$

$$\Gamma_A(\alpha) \in [0,1] \qquad (18)$$

c) $\exists \alpha_A$ (then is one and only one α_A), such that

$$\Gamma_A(\alpha_A) = 1 \;,\; \forall A \leqslant \wp \qquad (19)$$

d) $\left.\begin{matrix}\forall\;, \alpha < \alpha_A \\ \forall\; A \leqslant \wp\end{matrix}\right\}, \Gamma_A(\alpha)$ is monotonously non decreasing (20)

e) $\left.\begin{matrix}\forall\; \alpha > \alpha_A \\ \forall\; A \leqslant \wp\end{matrix}\right\}, \Gamma_A(\alpha)$ is monotonously non increasing (21)

f) A distance d is imposed in X such that:

$d(x_i, x_j) = \alpha_A$ (refer to c)

$\Gamma_A(\alpha_A) = 1$ and $\mathcal{R}(x_i, x_j) = \Gamma_A(\alpha)$ (22)

g) A composition operation is defined as follows:

$$\mathcal{R}(x_i, x_j) \oplus \mathcal{R}(x_j, x_k) = \Gamma_A(\alpha) \oplus \Gamma_B(\alpha) = \Gamma_{AB}(\alpha)$$

with the following properties:

g1) $\exists \alpha_{AB} \;,\; \Gamma_{AB}(\alpha_{AB}) = 1$ (23)

g2) $\Gamma_{AB}\left(\frac{\alpha}{\alpha_{AB}}\right) \leqslant \Gamma_C\left(\frac{\alpha}{\alpha_C}\right)$ (24)
when $\Gamma_C(\alpha_C) = 1$ and $\Gamma_C(\alpha) = \mathcal{R}(x_i, x_k)$

g3) $\forall \alpha < \alpha_{AB} \;,\; \Gamma_{AB}(\alpha)$ is monotonously non decreasing (25)

g4) $\forall \alpha > \alpha_{AB} \;,\; \Gamma_{AB}(\alpha)$ is monotonously non increasing (26)

Note: that the composition operation $\oplus$ is similar to a convolution.

C) Proximities using distribuition functions

$\wp$ is a family of distribuition functions as for instance the Gaussian probability density function:

a) $(x_i, x_j) \overset{\mathcal{A}}{\longmapsto} D_{ij} \in \wp$ (27)

$\mathcal{A}$ is functional in the sense that only one distribuition or function is invocated.

b) A convolution operation is introduced $\mathcal{A}(x_i, x_j) \circ \mathcal{A}(x_j, x_k) =$
$= D_{ij} \circ D_{jk} = D_{ijk}$ (28)

satisfying the following conditions:

$b_1)$ $\overline{D_{ijk}} \geqslant \overline{D_{ik}}$ (mean) (29)

$b_2)$ $\sigma(D_{ijk}) \geqslant \sigma(D_{ik})$ (variance) (30)

$b_3)$ $D_{ij} = D_{ji}$ (31)

$b_4)$ $D_{ii} = 0$, everywhere (32)

Note: that no distance is explicitly introduced but rules $b_1)$, $b_2)$, $b_3)$ and $b_4)$ are very similar.

"

The examples A, B and C show the versatility of the "proximity" concept, which is adaptable to real-case problems when the topological distance (Norm, Metrisation. etc.) cannot be applied.
"Proximities" induce in X quite loose structures but in the background there is a "distance" lurking.

IV) Application of Proximities

Some applications of proximities have been made, namely:

a) Thermodynamics:

The state of thermodynamic system can be described in a n-dimensional vector space, but no distance can be imposed in that space, the distance between two states has no physical sense.
The proximity imposed is the following: (3)

1) $X = \underset{i}{\times} \mathbb{R}^{i}, \quad \mathbb{R}^{i}$ real numbers (33)

2) $\wp \equiv \{ [q, \infty) : q \in \mathbb{R} \}$ (34)

3) $\forall_{\alpha,\beta}, x_\alpha, x_\beta \in X, \; (x_\alpha, x_\beta) \xrightarrow{\mathfrak{R}} [q_{\alpha\beta}, +\infty) \in \wp$
and $\mathfrak{R}$ is functional. (35)

4) $\forall_{\alpha,\beta}, \; \begin{cases} q_{\alpha\beta} = -q_{\alpha\beta} \\ q_{\alpha\alpha} = 0 \end{cases}$ (antisimetric) (36)

5) $\forall_{\alpha,\beta,\gamma}, \; q_{\alpha\beta} + q_{\beta\gamma} + q_{\gamma\alpha} = 0$ (37)

If $q_{\alpha\beta}$ is interpreted as the variation of the entropy in a reversible thermodynamic transformation starting in x_α and finishing in x_β, than:
The entropy variation ΔS actually accuring in real irreversible transformation starting in x_α and finishing in x_β is given by

$$\Delta S \in [q_{\alpha\beta}, \infty) \quad (38)$$

and the "produced" entropy in the real process is:

$$\delta S \in [0, \infty), \; \forall (x_\alpha, x_\beta) \quad \text{and} \quad \alpha \neq \beta \quad (39)$$

b) Classical Mechanics

Rigid rods systems with slack-joints can be studied, using slack--distances (III A). (2)

X vectorial space

$$\wp \equiv \{ [h_1, h_2] : h_1, h_2 \in R^+ \} \quad (40)$$

$$\mathfrak{R}(x_i, x_j) = [h_1^{ij}, h_2], \; \forall x_i, x_j \in X \quad (41)$$

$\mathcal{R}$ is functional and:

a) $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i)$ (42)

b) $h_{ij}^{1} + h_{jk}^{1} \geqslant h_{ik}^{1}$ and $h_{ij}^{2} + h_{jk}^{2} \geqslant h_{ik}^{2}$ (43)

c) $(h_{ij}^{2} - h_{ij}^{1}) + (h_{jk}^{2} - h_{jk}^{1}) \geqslant (h_{ik}^{2} - h_{ik}^{1})$ (44)

d) $[\mathcal{R}(x_i, x_j) \equiv \phi] \equiv [x_i \equiv x_j]$ (45)

c) Hypergraphs

If $\mathcal{E} \equiv (E_i \,;\, i \in I \text{ and } E_i \neq \phi)$. (46)

$\bigcup_{i \in I} E_i \equiv X$ and Card X is finite (47)

than $G(X, \mathcal{E})$ is a Hypergraph.

- A fuzzy-proximity can be imposed on $\mathcal{E}$, as, for instance, the one described in III B, where

$$\mathcal{R}(E_i, E_j) = \Gamma_{ij}(\alpha) \quad , \quad \forall E_i, E_j \in \mathcal{E} \quad (48)$$

- or a distribuition-proximity of the type IIIc) can readily be used, where:

$$\mathcal{R}(E_i, E_j) = D_{ij} \quad , \forall E_i, E_j \in \mathcal{E} \quad (49)$$

Both types of proximities, with the associated composition-operation, can be used as a figure of merit to choose different paths, connecting two vertices of X belonging to different $E_i, E_j \in \mathcal{E}$.

SHORT LIST OF KEY-WORDS

Proximity

Prox

Slack-proximity

Slack-prox

Fuzzy-sets proximities

Fuzzy-sets

Distribuition-proximity

Dist-prox

BRÈVE LISTE DE MOTS-CLÉS

Proximité

Prox

Proximité-intervale

Prox-int

Proximités-floues

Prox-floue

Proximité-distribution

Prox-dist

SHORT BIOGRAPHY

Mechanical Engineer; Professor at I.S.T. (Portuguese Technical University); now lecturing Automation (Systems).