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THE NON-LINEAR PROBLEM

«TÉCNICA»
REVISTA DE ENGENHARIA
Separata do n.º 358
Pág. 425 a 433

LISBOA
ASSOCIAÇÃO DOS ESTUDANTES
DO
INSTITUTO SUPERIOR TÉCNICO
1966

C. D. U. 531.395:518.5

THE NON-LINEAR PROBLEM

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RESUMO

O método proposto tem interesse quando as forças externas actuando no sistema mecânico, variam aleatòriamente, as ligações são descritas por funções não-lineares e discontínuas e finalmente, o objectivo do estudo é a detecção de acontecimentos cuja probabilidade é por vezes bastante pequena.

SYNOPSIS

The method proposed is of interest when the external forces acting on the mechanical system are changing randomly, the bounds are described by discontinuous non-linear functions and finaly, the aim of the study is the detection of occurances the probability of which is sometimes quite small.

● *Prof. Eng.º Vasco Costa, ao estudar problemas de amarração de navios de grande tonelagem, propôs um modelo físico de certa complexidade para explicar cabalmente os fenómenos observados.*

Porque problemas de natureza idêntica também ocorrem em engenharia mecânica, o referido Professor tomou a iniciativa de dar a oportunidade ao autor de procurar uma solução formal.

Simultâneamente uma solução experimental foi investigada pelo Eng.º Neves da Silva, ainda não publicada.

Um exemplo foi calculado no Centro de Cálculo da Fundação Calouste Gulbenkian.

O método e o exemplo foram apresentados numa das sessões da «Nato Advanced Study Institute» realizadas em Julho de 1965 e em Lisboa, e que foram de iniciativa do Prof. Vasco Costa.

Apresentamos neste número da Técnica *apenas o método.*

Desejamos deixar aqui exarados os nossos agradecimentos:

— *ao Prof. Vasco Costa, que está na origem deste artigo, pelo apoio e entusiasmo que emprestou.*
— *à Direcção do Centro de Cálculo da Fundação C. G. pelas facilidades que deu na execução do cálculo.*
— *aos doutores M. V. Cadete e A. Cadete por todo o trabalho de programação que é uma espinhosa tarefa, e pela paciência revelada na interpretação do método.*

0) INTRODUCTION

It is rather involved to solve formally problems regarding dynamical systems representable in a finite-dimensional phase-space by coordinates x^k and the corresponding momenta $m_{jk}\dot{x}^k$, where $\dot{x}^k$ is the time derivative of x^k, m_{jk} the generalised mass associated to x^k and j, $k = 1, \ldots, N$, if the elastic and viscous bounds are given by discontinuous, non-linear functions of x^k, $\dot{x}^k$ and time t.

The following treatment, adaptable to mechanical computation, provides a fairly general solution to such problems and is based on the following considerations:

a) It is easy to establish a recurrance formula to determine the state (x^k, $m_{jk}\dot{x}^k$) in time t_{i+1}, given the state in time t_i, where de lapse of time ($t_{i+1} - t_i$) is physicaly small (mathematicaly infinitesimal), and the acting forces are known.

b) If the number of iterations n resulting from successive application of the recurrance formula is not unduly large, the state of the system in time t_n is not unduly affected by computational errors.

c) If the choice of each initial state ($i = 0$) is made randomly and many such initial states are chosen, the coverage of the phase-space to be tested is randomly uniform.

d) Each iteration sequence initiated at a certain point of the phase-space may be considered as a truncated «histcry» of a dynamical experiment.

e) When a suficient number of the above refered «histories» of dynamical experiments have been obtained, the collection thus formed will be apreciated by a suitable criterium vector. Each element of the criterium vector corresponds to a previously well defined question or measure, and each vector corresponds to a distinct experiment.

f) A statistical analysis can be made of the collection of vectors obtained from the collection of experiments.
The collection of vectors can be considered a statistic based on a small (or large) sample of the Universe under scrutinity.

g) If, by statistical procedure, the sample is found to be small, the same sample can be allways enlarged by randomly choosing new starting points and producing new dynamical experiments.

h) Finaly, if some region of the phase-space is considered of particular interest, the random choice of initial states, can be conditioned to that region and a finer scrutinity of the region can be thus obtained.

The method proposed is of particular interest:

— When the external forces acting on the systems are of a changing nature and a steady-state is never reached by the system.
— When the bounds (elastic and viscous) are described by discontinuous non-linear functions.
— When the aim of the study is the detection of rather improbable occurances or the evaluation of the probability of occurrance of certain types of phenomena, and not a description of the mouvement as such.

A formalised sumary of the method is presented in Chapter V.

I) THE MATHEMATICAL MODEL

The dynamical state of a discrete system in an Eucleadean Space and absolute time can be fully described by N parameters, their N respective time derivatives and time.

Representing by:

x^k $(k = 1,\ldots N)$	The generalised parameter
$\dot{x}^k$ $(k = 1,\ldots N)$	Its first time derivative
A_{kj} $(k,k = 1,\ldots N)$	The inertial Matrix supposed to be always invertible
R_j $(j = 1,\ldots N)$	A generalised viscous force
f_j $(j = 1,\ldots N)$	A generalised elastic force
F_j $(j = 1,\ldots N)$	A generalised external force

It is true, at any time t_i, that:

1.1
$$\lim_{\delta t_i \to 0} [A_{kj}(t_i + \delta t_i) \cdot \dot{x}^k (t_i + \delta t_i) - A_{kj}(t_i) \cdot \dot{x}_k (t_i)] = \lim_{\delta t_i \to 0} [R_j(t_i) + f_j(t_i) + F_j(t_i)] \cdot \delta t_i$$

Which is a straightforward application of the formula $\delta (M \cdot \dot{x}) = F \cdot \delta t$.
Defining $\delta A_{kj} (t_i)$ as:

$$\delta A_{kj} (t_i) = A_{kj} (t_i + \delta t_i) - A_{kj} (t_i)$$

the first member of the equation 1.1. can be restated as follows:

$$\lim_{\delta t_i \to 0} \left\{ A_{kj} (t_i) [\dot{x}^k_1 (t_i + \delta t_i) - \dot{x}^k (t_i)] + \delta A_{kj} (t_i) \cdot \dot{x}^k (t_i + \delta t_i) \right\}$$

If $\lim_{\delta t_i \to 0} \delta A_{kj} (t_i)$ is an infinitesimal quantity of greater order than $\lim_{\delta t_i \to 0} [x_k (t_i + \delta t_i) - \dot{x}^k (t)]$ then $\delta A_{kj} (t_i) \cdot x^k (t_i + \delta t_i)]$ can be negleted and the first member of the equation takes the reduced form of:

$$\lim_{\delta t_i \to 0} \left\{ A_{kj} (t_i) [\dot{x}^k (t_i + \delta t_i) - \dot{x}^k (t_i)] \right\}$$

Physically, this implies that $A_{kj} (t_i)$ varies slowly with time compared with $\dot{x}^k$ variation with time.
We will use the simplified expression 1.1' to describe the method:

$$1.1') \quad \lim_{\delta t_i \to 0} A_{kj} [\dot{x}_k (t_i + \delta t_i) - \dot{x}_k (t_i)] = \lim_{\delta t_i \to 0} [\bar{R}_j (t_i) + \bar{f}_j (t_i) + \bar{F}_j (t_i)] \cdot \delta t_i$$

where:

a) $\bar{R}_j$, $\bar{f}_j$ and $\bar{F}_j$ are averages of R_j, f_j and F_t computed in the interval δt_i, as for instance, with a Stieltjes measure:

$$\frac{\oint_{t_i}^{t_i + \delta t_i} R_j \, dt}{\oint_{t_i}^{t_i + \delta t_i} dt} = \bar{R}_i$$

$\oint$ is the symbol for the Stieltjes integral

b) $\delta t_i = t_{i+1} - t_i$ and, in general, $\delta t_i = \delta t_k$ for all i and k, with: i , $|k| = 0,1 \ldots N$.

c) A_{ki} is invariant (or slowly variable with time) and invertible.

The expression (1.1') can be presented in the following form:

$$1.2) \quad \lim_{\delta t_i \to 0} [\dot{x}^k (t_i + \delta t_i) - \dot{x}^k (t_i)] = \bar{\bar{A}}^{-1}_{kj} \lim_{t_i \to 0} (\bar{R}_j + \bar{f}_j + F_i) \cdot \delta t_i$$

(1.2) permits the computation of $\dot{x}_k$, in time $t_i + \delta t_i$ if $\dot{x}^k$ in time t_i, is known.

The computation of x^k is readily made:

$$1.3) \quad x^k (t_i + \delta t_i) = x^k (t_i) + \frac{\dot{x}^k (t_i + \delta t^i) + \dot{x}^k (t_i)}{2} \cdot \delta t_i$$

If the initial state is given (i = o), $\left[x_k (t_o), \dot{x}^k (t_o) \right]$, and all the acting forces are known, it is possible, using formulas 1.2 and 1.3 to compute, step — wise, the state of the dynamical system.
The interval δt_i should be kept small to reduce errors which accumulate as i grows.

II) DESCRIPTION OF THE FUNCTIONS F, f, R.

The functions f, F and R may take many forms and the functions presented in Chapter II), have to be considered as typical exemples, used to describe the method and to show its power and flexibility.

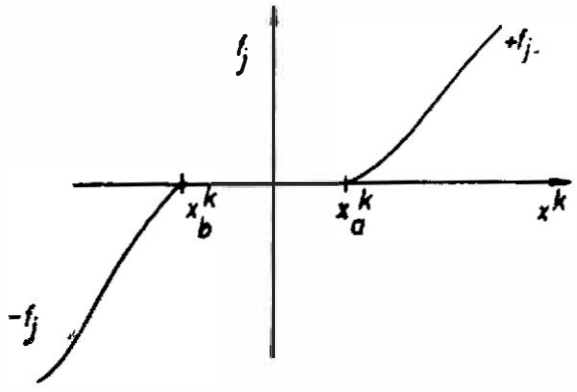
1) *Function* F_j:

F_j depends on time (t_i) only, and has the general form of:

2.1) $$F_j(t_i) = \sum_{r=1}^{n} G_r \cos(\omega_r t_i + \alpha_{rj})$$

$$n = 1, \ldots [\infty$$

The formula 2.1) is sufficiently general to describe the *external forces* encountered in most of the real cases.



2) *Function* f_j

Function f_j will be described in three parts or intervals:

$$x^k > x^k_a \longrightarrow f_j(x,t) = + f_j[(x^k - x^k_a), t]$$

$$x^k_b \leq x^k \leq x^k_a \longrightarrow f_j(x,t) = 0 \qquad 2.2)$$

$$x^k < x^k_b \longrightarrow f_j(x,t) = -f_j[(x^k - x^k_b), t]$$

with $x^k_a > x^k_b$

The above set of conditions implies that it is given:

a) a functional Matrix $\{+ f_j[(x^k - x^k_a), t]\}$ 2.3)

$j,k = 1 \ldots N$, with N^2 functions $+ f_j[(x^k - x^k_a), t]$

b) a functional Matrix $\{-f_j[(x^k - x^k_b), t]\}$ 2.4)

again with N^2 functional elements.

c) The functions,

$$\begin{cases} x^k_a = x^k_a(x^1, \ldots, x^N) \\ x^k_b = x^k_b(x^1, \ldots, x^N) \end{cases} \qquad k = 1, \ldots, N$$

that is, a functional Matrix of the form:

$$[x^k_a(x^i)] \qquad 2.5)$$

$x^k_a - x^k_b$ may be called the *clearance*, and $x^k_a > x^k_b$.

Although any other description could have been chosen, it was felt that for $+f_j$ and $-f_j$, a function with the following form could be adopted:

$$+f_j\,[(x^k - x_a^k),\ t] = \sum_{r=1}^{p} a_r\,(x^k - x_a^k)^r \qquad 2.6)$$

$$-f_j\,[(x^k - x_a^k),\ t] = \sum_{r=1}^{q} b_r\,(x^k - x_b^k)_r \qquad 2.7)$$

(Polynomial form)

with $r = 1, \ldots [\infty$.

Regarding $[x_a^k\,(x^j)]$, the expression has to be established in each case according to the geometry of the problem.

3) *Function* R_j:

Again R_j can be treated as f_j.

The choice of $R^j\,(\dot{x})$ is made in accordance with each problem.

Note 1: Not knowing the geometry of the System, the following expression may be adopted instead:

$$x_a^k = \sum_{l=1}^{m} H_l \cos\,(\omega_l\,t + \alpha_{Kl}) \qquad 2.8)$$

$$x_b^k = \sum_{g=1}^{s} H_g \cos\,(\omega_g\,t + \alpha_{gk}) \qquad 2.9)$$

with $m, s, = 1, \ldots [\infty$.

III) DESCRIPTION OF THE COMPUTATIONAL METHOD

The mathematical model of the dynamical System was described and it was shown how to compute the state of the System in time t_{i+1}, if it is known the state in time t_i and the applied forces during the time internal $(t_{i+1} - t_i)$.

Chapter III. is devoted to describe:

a) The choice of the initial state
b) The criterium-vector
c) The statistical treatment of the data collected

a) *The choice of the initial state*

a_1) Each problem is defined by 4 sets of parameters with relation to:

The dynamical sistem:

$$\begin{array}{llll} & A_{kj} & & \\ & +f & {}_0f & -f \\ 3.1) & +R & {}_0R & -R \\ & D_{máx}^k = máx & |x^k - x_a^k| & \text{with } x^k > x_a^k \\ & d_{máx}^k = máx & |x^k - x_b^k| & \text{with } x^k < x_b^k \end{array}$$

The external Force:

$$3.2) \qquad \begin{cases} G_r \\ {}_0r \end{cases} \qquad r = 1, \ldots n$$

The clearence (x_a^k *and* x_b^k) :

Taking in account Note 1, x_b^k and x_b^k are defined by :

3.3)
$$\begin{cases} \left\{ \begin{matrix} H_l \\ \omega_l \end{matrix} \right\} 2.8 & l = 1, \ldots m \\ \left\{ \begin{matrix} H_g \\ \omega_g \end{matrix} \right\} 2.9 & g = 1, \ldots s \end{cases}$$

The remaining parameters, α_{rj}, α_{kl}, α_{kg} and $x_k(t_o)$. $\dot{x}^k(t_o)$ are treated as random variables defining the initial state.

a_2) The computer memorises all the information referred in 3.1, 3.2 and the randomly obtained T sets of α_{rj}, α_{kl}, α_{kg}, $x^k(t_o)$ and $\dot{x}^k(t_o)$ are punched beforehand and the computer memorises one set at a time and runs a test and prints out the *programmed information* (See : criterium vector b)) and proceeds to the next set, till all the T sets are processed.

a_3) The random variables above refered to, are chosen in the following intervals :

α_{rj} α_{kl} α_{kg} : in the interval $(0,2\pi)$

$\left.\begin{matrix} x^k(t_o) \\ \dot{x}^k(t_o) \end{matrix}\right\}$ in the interval to be tested

a_4) One important factor is the counting of the cycles. This is made by verification of a succession of the following ordered pair of inequalities :

3.4)
$$x^k(t_i) < x_b^k(t_i + r)$$
and
$$x^k(t_i + r) > x_b^k(t_i + r)$$

b) *The criterium vector*

To exemplify a criterium vector, let us admit that the problem consists in studying the breakage of the elastic bounds and it is assumed that this ocurrance takes place when their deformation is greater than a certain predetermined limit.

The computer is programmed to stop each run when one of the three occurences is fulfilled :

1) $$\left| x^k(t_i) - x_a^k(t_i) \right| > D_{max}^k \qquad 3.5)$$
$$(x^k > x_a^k)$$

or

2) $$\left| x^k(t_i) - x_b^k(t_i) \right| > d_{max}^k \qquad 3.6)$$
$$(x^k < x_b^k)$$

or

3) The number of cycles exceeds L, where L is a predefined constant.

At the and of each run the computer prints out the following information :

— What was the reason [1), 2) ou 3)] that caused the stoppage

— If the third reason, the max $\left| x^k - x_a^k \right|$ and max $\left| x^k - x_b^k \right|$

and the times (t_i, t_j) when these maxima have taken place.

c) *The statistical treatment of the data collected*

After T runs, the T sets of random variables have been successively introduced and sets of information were printed out (or punched out).

The next problem is to treat the date obtained by statistical method and to study the results. This is a straightforward operation which is not described here.

Final remarks

Now the question arises why not simply run the computer with some pre-set data till some kind of steady-state regime is attained.

The reasons are:

- The errors of computational origin would accumulate. It is better to start with fresh random variables and run short trials.
- A steady-state may never be attained, as the external forces are constantly changing.
- The coverage of the unknown universe is more uniform, recurring to random sampling.
- The system may have more than one stable modes of vibration disconnected between themselves due to the nonlinearity of the bounds and in that case from one region the other could never be attained.

IV) APPLICATION TO A ONE DIMENSIONAL PROBLEM

The feasibility of the method can be tested with a one-dimensional problem and this chapter was written to this effect.

To reduce the number of steps, that is, to increase δt_i without unduly increasing the computational errors, the following alterations were introduced.

4.1)
$$\dot{x}^k(t_{i+1}) = \dot{x}^k(t_i) + \ddot{x}^k(t_i) \cdot \delta t_i$$

$$\text{but } \ddot{x}^k(t_i) = \frac{\dot{x}^k(t_i) - \dot{x}^k(t_{i-1})}{\delta t_i}$$

$$\dot{x}^k(t_{i+1}) = 2\dot{x}^k(t_i) - \dot{x}^k(t_{i-1})$$

4.2)
$$\frac{\dot{x}^k(t_{i+1}) + \dot{x}^k(t_i)}{2} = \frac{1}{2}\left(3\dot{x}^k(t_i) - \dot{x}^k(t_{i-1})\right)$$

4.3)
$$x^k(t_{i+1}) = x^k(t_i) + \frac{\dot{x}^k(t_{i-1}) - \dot{x}^k(t_i)}{2} \cdot \delta t_i$$

$$= x^k(t_i) + \left(3\dot{x}^k(t_i) - \dot{x}^k(t_{i-1})\right)\frac{\delta t_i}{2}$$

4.4)
$$\bar{f}_j(t_i) = \frac{1}{2}\left(f_j(t_i) + f_j(t_{i+1})\right)$$

4.5)
$$\bar{R}_j(t_i) = \frac{1}{2}\left(R_j(t_i) + R_j(t_{i+1})\right)$$

4.6)
$$\bar{F}_j(t_i) = \frac{1}{2}\left(F_j(t_i) + F_j(t_{i+1})\right)$$

The underlined symbols have the following significance:
If $S(t_i)$ is a symbol than

$$\lim_{\delta t_i \to 0} \underline{S}(t_i) \to S(t_i)$$

The *overlined* symbols mean averages (in this case linear averages).
The interval δt_i was established in such a way that, after L cycles, the results with δt_i, $2.\delta t_i$, $\frac{1}{2}\delta t_i$ were equivalent, that is, the first two figures matched.

V) FORMALISED SUMARY OF THE METOHD

1) The dynamical state of a mechanical system is representable in the phase-space by the ordered pair of vectors $[x^k(t), A_{jk}(t)\,\dot{\underline{x}}^k(t)]$

Where:

$x^k(t)$ is the generalised coordinate

$A_{jk}(t)\,\dot{x}(t)$ is the generalised momentum

and

$A_{jk}(t)$ is the inertial matrix, allways invertible

$\dot{x}^k(t)$ is the time derivative of $x^k(t)$

t is the absolute time

$j, k = 1, \ldots N$, the N coordinates, with $N < \infty$

The applied force to the inertial system is symbolised by $\Psi_j(t)$ and $j = 1, \ldots N < \infty$

We note that $\Psi_j(t) \equiv \Psi_j[x(t), \dot{x}(t), t]$

2) The momentum conservation law, in its generalised form, can be described by the following expression:

$$\delta\,[A_{jk}(t) \cdot \dot{x}^k(t)] = \Psi_j(t) \cdot \delta t \quad \ldots \quad a)$$

It is also true that: $x^k(t) = \dot{x}^k \cdot \delta t \quad \ldots \quad b)$

Conjugating expressions a) and b), it is possible to evaluate state

$$[x^k(t + \delta t),\ A_{jk}(t + \delta t)\ \dot{x}^k(t + \delta t)]$$

if it is given:

$$[x^k(t),\ A_{kj}(t)\ x(t)] \quad \text{and} \quad \Psi_j(t + \delta t),\ \Psi_j(t),\ A_{jk}(t + \delta t),\ A_{jk}(t).$$

3) Generalising the application of a) and b) successively, the general operator O_R establishes the following relation:

$$[x^k(t + R.\,\delta t),\ A_{jk}(t + R\delta t).\ \dot{x}_k(x + R\delta t)] =$$
$$= O_R[(x^k(t),\ A_{jk}(t).x^k(x), \psi_j(t), \ldots, \psi_j(t + R.\delta t), A_{jk}(t), \ldots, A_{jk}(t + R.\,\delta t)] \quad \ldots \quad c)$$

and $R = 0,1,2,\ldots S$ and S is given a priori.

4) Randomly, τ initial states are chosen.
Each initial state is described by a pair of vectors

$$P_{s,o} = [x^k(t), A_{jk}(t).\dot{x}^k(t)]$$

and $s = 1, \ldots \tau < \infty$

Using repeatedly the operator O_R, S pairs $P_{s,R} = [x^k(t + R\delta t), A_{jk}(t + R.\delta t), x^k(t + R.\delta t)]$ are computed out of each initial state $P_{s,0}$ and the set $P_s = \{P_{s,R} : R = 0,1,\ldots S\}$ is formed

Finaly the collection of sets $\pi = \{P_s : s = 1,2,\ldots \sigma\}$

contains σ sets P_s each containing $S + 1$ pairs $P_{s,R}$.

We shall denote by:

— Experiment, each set P_s

— Story of the experiment (s), the $S + 1$ pairs $P_{s,R}$

— Sample, the collection π of σ experiments.

5) *A criterium vector* $\{\mathcal{S}_v = \mathcal{S}_v : v = 1,\ldots V\}$ is chosen in accordance to the particular problem to be solved.

Each element $\mathcal{S}_v \in \{\mathcal{S}_v\}$ is *a measure* defined on the set $U\pi$, of all samples that can be taken out of the Universe of pairs $[x^k, A_{jk}\dot{x}^k]$.

No restriction is made regarding the nature of the measures $\mathcal{S}_v$, outside their definition on the set $U\pi$, they can be so simple as a binary measure (true or false with respect to a certain proposition) or discrete or dense measure.

6) Each pair $P_{s,R} \in \pi$ is measured by means of the criterium-vector $\{\mathcal{S}_v\}$ and the corresponding measure-vector E_{vsR} is evaluated.

The set $E_{v,s} = \{E_{vsR} : R = 0,1,\ldots S\}$ is the set of measure-vectors of the set P_s.

The collection of sets $E_v = \{E_{x,s} : s = 1,\ldots \sigma\}$ is the collection of sets of measures-vectors of the sample.

The sample π is now fully and completely measured by means of an uniform criterium-vector $\{\mathcal{S}_v\}$ and a collection of sets E_v of mesure-vectors was obtained.

7) As the sample of the initial states was randomly obtained, statistical methodology can be correcty applied to E_v.

This last operation is not described here, as it is quite straightforward and a standard method.